Analytic description of atomic interaction at ultracold temperatures. II. Scattering around a magnetic Feshbach resonance

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(Received 31 May 2011; published 8 August 2011)

Starting from a multichannel quantum-defect theory, we derive analytic descriptions of a magnetic Feshbach resonance in an arbitrary partial wave l and the atomic interactions around it. An analytic formula, applicable to both broad and narrow resonances of arbitrary l, is presented for ultracold atomic scattering around a Feshbach resonance. Other related issues addressed include (a) the parametrization of a magnetic Feshbach resonance of arbitrary l, (b) rigorous definitions of "broad" and "narrow" resonances of arbitrary l and their different scattering characteristics, and (c) the tuning of the effective range and the generalized effective range by a magnetic field.

DOI: 10.1103/PhysRevA.84.022706 PACS number(s): 34.10.+x, 34.50.Cx, 33.15.−e, 03.75.Nt

I. INTRODUCTION

Analytic descriptions of two-body interactions are highly desirable if any systematic understanding of quantum few-body and quantum many-body systems is to be expected or achieved. The best-known example may be the Gross-Pitaevskii theory of identical bosons [1], with its simplicity and generality depending intimately on our ability to parametrize the low-energy two-body interaction using the effective range theory (ERT) [2–4]. The same is true for quantum few-body theories in the universal regime (see, e.g., Refs. [5,6]).

In a companion paper [7], referred to hereafter as paper I, we have discussed the limitations of the standard ERT [2–4] in describing atomic interactions at low temperatures and how such limitations are overcome using expansions derived from the quantum-defect theory (QDT) for $-1/r^6$ type of long-range potentials [8–10]. The focus was on the case of a single channel, both out of the necessity of theoretical development and to provide a set of single-channel universal behaviors that will serve as benchmarks for understanding other types of behaviors.

This article extends this discussion to atomic interaction around a magnetic Feshbach resonance [11–13] in an arbitrary partial wave l. It is a nontrivial extension with considerable new physics as a Feshbach resonance is necessarily a multichannel phenomenon [11–13], for which few analytic results have been derived in any general context. The theory includes the parametrization of the resonance, the rigorous definitions of "broad" and "narrow" resonances [12–15], and an analytic description of the atomic scattering properties around them. Such understandings, which have been mostly limited to the s wave [12,13], are not only of interest by themselves, they are also prerequisites for understanding atomic interaction in an optical lattice [16] and behaviors of quantum few-atom and many-atom systems around a Feshbach resonance. For nonzero partial waves, the theory here is a necessity as ERT fails [7,8,17,18]. Even for the s wave, it offers much improved analytic description, especially for narrow resonances around which the energy dependence of the scattering amplitude can become so significant that it has to be incorporated into the corresponding few-body [19] and many-body theories (See, e.g., Refs. [20–23]).

There are three main steps in developing an analytic description of a magnetic Feshbach resonance. The first is the reduction of the underlying multichannel problem, as formulated in a multichannel quantum-defect theory (MQDT) of Ref. [24] to an effective single-channel problem. The second is an efficient parametrization of a magnetic Feshbach resonance. The third is to apply the theory of paper I [7] to obtain the desired results such as the scattering properties around the threshold, to be addressed in this article.

The paper is organized as follows. The reduction to an effective single-channel problem is carried out in Sec. II. The parametrization of a magnetic Feshbach resonance is addressed in Sec. III. In particular, we derive in Sec. III B the magnetic-field dependence of scattering lengths and the generalized scattering lengths introduced in paper I [7]. We show in this section that regardless of l, the scattering length, or the generalized scattering length for $l \geqslant 2$, can be parametrized around a magnetic Feshbach resonance in a similar fashion as the s wave scattering length [12,13,25]. The parametrization is further developed in Sec. III C in terms of scaled parameters. It leads not only to more concise analytic formulas but, more importantly, to rigorous definitions of "broad" and "narrow" Feshbach resonances of arbitrary l. In Sec. IV, we present the QDT expansion [7] that provides an analytic description of ultracold scattering around a magnetic Feshbach resonance of arbitrary l. As sample applications of the QDT expansion, Sec. V presents and discusses the generalized effective range expansion [7] for ultracold scattering around a magnetic Feshbach resonance. It includes a relationship between the (generalized) effective range and the (generalized) scattering length that is applicable to both broad and narrow resonances and resonances of arbitrary l. It substantially extends a previous relationship [7,8,20,26] that is applicable only to broad resonances. Two special cases of interest in cold-atom physics, the case of infinite scattering length (the unitarity limit) and the case of zero scattering length, are also discussed in this section as examples of the QDT expansion. The conclusions are given in Sec. VI.

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II. REDUCTION OF A MULTICHANNEL PROBLEM TO AN EFFECTIVE SINGLE-CHANNEL PROBLEM

In cold-atom physics, most of the interest in atomic interaction lies in a small range of energies around the lowest threshold (of a certain symmetry), below which we have true bound states. Ignoring weak couplings between different partial waves due to the magnetic dipole-dipole [25,27] and the second-order spin-orbit interaction [28–30], we can label the single channel of partial wave l that is associated with this lowest threshold "a" and all the other channels of partial wave l by "c." Above the lowest threshold and below the energies at which the second or more channels becomes open, it is already clear from Ref. [24] that the MQDT for atom-atom interaction reduces to an effective single-channel problem with an effective short-range K-matrix, K^c, given by

$$K^c_{\rm eff} = K^c_{aa} + K^c_{ac}\left(\chi^c_{cc} - K^c_{cc}\right)^{-1}K^c_{ca}. \tag{1}$$

Here, K^c_{aa}, K^c_{ac}, K^c_{ca}, K^c_{cc}, are submatrices of K^c corresponding to the separation of all channels into a single "a" channel and N_c closed "c" channels. χ^c_{cc} is an $N_c \times N_c$ diagonal matrix with elements $\chi^c_l(\epsilon_{si})$, which is the universal χ^c_l function, as given, e.g., by Eq. (54) in paper I, evaluated at properly scaled channel energies. This reduction to an effective single-channel problem is a result of the standard channel-closing procedure, and occurs in similar fashion in any type of multichannel scattering theories. What is important, and maybe less well-known, is that the energies of the multichannel bound states below the threshold "a" can also be reduced to an effective single-channel problem with the *very same* effective K^c as given by Eq. (1). A proof is given in the Appendix A.

With this reduction, the scattering below the second threshold and the multichannel bound states below the threshold "a" are all described by an effective single-channel QDT [8–10] with an effective short-range K^c matrix given by $K^c_{\rm eff}$. Specifically,

$$K_l = \tan\delta_l = \left(Z^c_{gc}K^c_{\rm eff} - Z^c_{fc}\right)\left(Z^c_{fs} - Z^c_{gs}K^c_{\rm eff}\right)^{-1} \tag{2}$$

gives the scattering K matrix between the lowest and the second thresholds, and the solutions of (see Appendix A)

$$\chi^c_l(\epsilon_s) = K^c_{\rm eff} \tag{3}$$

give the bound spectrum below the lowest threshold. Here, $Z^c_{xy}(\epsilon_s,l)$ are universal QDT functions for $-1/r^6$ potential, as given, e.g., by Eqs. (4)–(7) of paper I. They, and $\chi^c_l(\epsilon_s)$, are all evaluated at a scaled energy relative to the lowest threshold of angular momentum l, $\epsilon_s = \epsilon/s_E = (E - E_a)/s_E$, with $s_E = (\hbar^2/2\mu)(1/\beta_6)^2$ being the energy scale and $\beta_6 = (2\mu C_6/\hbar^2)^{1/4}$ being the length scale associated with the $-C_6/r^6$ van der Waals interaction in channel "a." We note that other than the ignorance of weak interactions that couple different l states, there is no further approximation associated with this reduction to a single channel.

This effective single-channel problem differs from a true single-channel problem in that the energy dependence of $K^c_{\rm eff}$ is generally not negligible, unlike the K^c parameter for a single channel [9]. As will become clear throughout this work, it is this energy dependence, which originates from the energy dependence of χ^c_{cc} in Eq. (1), that leads to deviations from single-channel universal behaviors of paper I [7] and makes the behaviors of a "narrow" Feshbach resonance differ substantially from those of a "broad" Feshbach resonance. As another difference from a true single-channel problem, the l dependence of χ^c_{cc} also makes $K^c_{\rm eff}$ l-dependent. The same formalism applies to atomic interaction in an external magnetic field [31], which has the additional effect of making $K^c_{\rm eff}$ depend parametrically on B. We will use the notation of $K^c_{\rm eff}(\epsilon,l,B)$, when necessary, to fully specify its dependences.

The equivalence, around the lowest threshold, of the multichannel atomic interaction in a B field to an effective single-channel problem with an effective short-range $K^c_{\rm eff}(\epsilon,l,B)$ makes most results of paper I [7] immediately applicable, except for a few that made explicit use of the energy- and/or l-insensitivity of K^c. In particular, if we define a K^{c0}_l parameter, which is more convenient for descriptions of near-threshold properties [7,10], as

$$K^{c0}_l(\epsilon,B) = \frac{K^c_{\rm eff}(\epsilon,l,B) - \tan(\pi\nu_0/2)}{1 + \tan(\pi\nu_0/2)K^c_{\rm eff}(\epsilon,l,B)}, \tag{4}$$

where $\nu_0 = (2l+1)/4$ for $-1/r^6$ type of potential, the locations of the zero-energy magnetic Feshbach resonances in an arbitrary partial wave l, B_{0l}, namely the magnetic fields corresponding to having a bound or quasibound state right at the threshold, can be conveniently found as the roots of $K^{c0}_l(\epsilon=0,B)$ [7,32], namely as the solutions of

$$K^{c0}_l(\epsilon=0,B_{0l}) = 0. \tag{5}$$

The scattering length or the generalized scattering length for an abitrary l is given by the zero-energy value of K^{c0}_l through Eq. (47) of paper I [7], namely,

$$\widetilde{a}_l(B) = \bar{a}_l\left[(-1)^l + \frac{1}{K^{c0}_l(\epsilon=0,B)}\right], \tag{6}$$

where $\bar{a}_l = \bar{a}_{sl}\beta_6^{2l+1}$ is the mean scattering length (with scale included) for partial wave l that was defined in paper I [7], with

$$\bar{a}_{sl} = \frac{\pi^2}{2^{4l+1}[\Gamma(l/2+1/4)\Gamma(l+3/2)]^2} \tag{7}$$

being the scaled mean scattering length. Recall that the generalized scattering length, $\widetilde{a}_l$, reduces to the regular scattering lengths whenever they are well defined, namely for the s and p partial waves.

Computationally, a similar theory based on MQDT [24] has been shown by Hanna *et al.* [31] to give an accurate description of magnetic Feshbach resonances over a wide range of magnetic fields using only three parameters for alkali-metal systems. Even better results can be expected by incorporating the energy and/or partial wave dependences of K^c_S and K^c_T [24] using a few more parameters. Further calculations for specific systems and especially nonzero partial waves will be presented elsewhere. Here we focus on the parametrization of one particular resonance and the analytic description of atomic interaction around it. Specifically, a parametrization of the energy and the magnetic field dependences of the K^{c0}_l parameter around a magnetic Feshbach resonance, to be developed in the next section, will enter into the description of the $K^{(D)}_l$ term of paper I [7], to describe the deviation of the phase shift from the Born term ($\sim k^4$) around the resonance.

III. PARAMETRIZATION OF A MAGNETIC FESHBACH RESONANCE

A. Derivation and general considerations

As the second step toward developing an analytic description of a magnetic Feshbach resonance, we need a simple parametrization of $K^c_{\rm eff}$ or the corresponding K^{c0}_l. For any isolated resonance, the second term in Eq. (1) has a simple pole at $\bar{\epsilon}_l(B)$, determined by $\det(\chi^c - K^c_{cc}) = 0$. It represents the "bare" location of a Feshbach resonance and depends on the magnetic field. (Here, "bare" means no coupling to the open channel "a.") Around such a simple pole, the effective K^c parameter, Eq. (1), can always be parametrized as

$$K^c_{\rm eff} = K^c_{{\rm bg}l} - \frac{\Gamma^c_l/2}{\epsilon - \bar{\epsilon}_l(B)}, \tag{8}$$

sufficiently close to the pole. Here, Γ^c_l is a measure of the width of the resonance, ϵ and $\bar{\epsilon}_l$ are energies that are conveniently chosen to be relative to channel "a," e.g., $\epsilon = E - E_a$, and $K^c_{{\rm bg}l}$ is a background K^c parameter, namely the K^c for energies and magnetic fields away from the resonance, such that $|\epsilon - \bar{\epsilon}_l(B)| \gg \Gamma^c_l$. Using the fact that χ^c_l is a piecewise monotonically decreasing functions of energy [9], namely, $d\chi^c_l/d\epsilon_s < 0$, one can further show rigorously that $\Gamma^c_l > 0$, a property that will put important constraints on other forms of parametrizations, all of which will be derived from Eq. (8).

In writing Eq. (8), we have adopted a notation that avoids unnecessary confusions without getting into the details of the MQDT [24] for atomic interaction in a magnetic field [31]. Rigorously speaking, the K^c matrix itself, and therefore the parameters $K^c_{{\rm bg}l}$ and Γ^c_l in Eq. (8), also depend on B. This dependence, however, is only significant over a field range of the order of $\Delta E^{\rm hf}/\mu_B$, where $\Delta E^{\rm hf}$ is the atomic hyperfine splitting, and μ_B is the Bohr magneton. Since our focus here is on the parametrization of an individual resonance, the width of which is always much smaller than the hyperfine splitting [13], we adopt the notation of Eq. (8) to emphasize that over the range of B field of interest here, the most relevant B field dependence is that of the "bare" Feshbach energy, $\bar{\epsilon}_l(B)$.

As discussed in paper I [7], analytic properties around the threshold are more conveniently described using the short-range parameter K^{c0}_l. Substituting Eq. (8) into Eq. (4), we have

$$K^{c0}_l(\epsilon,B) = K^{c0}_{{\rm bg}l} - \frac{\Gamma^{c0}_l/2}{\epsilon - \bar{\epsilon}_l(B) - f_{El}}, \tag{9}$$

where $K^{c0}_{{\rm bg}l}$ is the background K^{c0}_l parameter corresponding to $K^c_{{\rm bg}l}$

$$K^{c0}_{{\rm bg}l} = \frac{K^c_{{\rm bg}l} - \tan(\pi\nu_0/2)}{1 + \tan(\pi\nu_0/2)K^c_{{\rm bg}l}}, \tag{10}$$

with a corresponding generalized background scattering length of [7]

$$\widetilde{a}_{{\rm bg}l} = \bar{a}_l\left[(-1)^l + \frac{1}{K^{c0}_{{\rm bg}l}}\right], \tag{11}$$

and

$$\Gamma^{c0}_l = \Gamma^c_l\frac{1 + \tan^2(\pi\nu_0/2)}{\left[1 + \tan(\pi\nu_0/2)K^c_{{\rm bg}l}\right]^2}. \tag{12}$$

The f_{El} in Eq. (9) is not an independent parameter. It is related to $K^{c0}_{{\rm bg}l}$ and Γ^{c0}_l by

$$f_{El} = \frac{1}{2}\Gamma^{c0}_l\frac{\tan(\pi\nu_0/2)}{1 - \tan(\pi\nu_0/2)K^{c0}_{{\rm bg}l}}. \tag{13}$$

In describing a Feshbach resonance in terms of K^{c0}_l, the fact that $\Gamma^c_l > 0$ translates into the condition of $\Gamma^{c0}_l > 0$, as is clear from Eq. (12).

From Eq. (9), the most general parametrization of a magnetic Feshbach resonance is that of Appendix B. We adopt here a slightly less general parametrization that uses parameters that have more direct physical interpretations and are more closely aligned with those already well established for the s wave [12,13,25].

For $K^{c0}_{{\rm bg}l} \neq 0$ ($\widetilde{a}_{{\rm bg}l} \neq \infty$), namely in all cases when there is no background bound or quasibound state right at the threshold [32], it is more convenient to rewrite Eq. (9) as

$$K^{c0}_l(\epsilon,B) = K^{c0}_{{\rm bg}l}\frac{\epsilon - \epsilon_l(B)}{\epsilon - \epsilon_l(B) - d_{El}}, \tag{14}$$

$$= K^{c0}_{{\rm bg}l}\left[1 + \frac{d_{El}}{\epsilon - \epsilon_l(B) - d_{El}}\right], \tag{15}$$

where

$$\epsilon_l(B) = \bar{\epsilon}_l(B) + \frac{\Gamma^c_l/2}{K^c_{{\rm bg}l} - \tan(\pi\nu_0/2)}, \tag{16}$$

and

$$d_{El} = (\Gamma^c_l/2)\frac{1 + \tan^2(\pi\nu_0/2)}{\left[\tan(\pi\nu_0/2) - K^c_{{\rm bg}l}\right]\left[1 + \tan(\pi\nu_0/2)K^c_{{\rm bg}l}\right]}. \tag{17}$$

In this form for K^{c0}_l, the location of the zero-energy magnetic Feshbach resonance, B_{0l}, determined by Eq. (5), translates into the solution of $\epsilon_l(B_{0l}) = 0$. And since we are interested here only in a range of B that covers a single Feshbach resonance, the $\epsilon_l(B)$ in Eq. (15) can be approximated, around B_{0l}, by $\epsilon_l(B) \approx \delta\mu_l(B - B_{0l})$, where $\delta\mu_l = d\epsilon_l(B)/dB|_{B=B_{0l}}$ is the difference of magnetic moments between the molecular state and the separate-atom state [13]. This approximation, together with Eq. (15), gives the following parametrization of the effective K^{c0}_l around a magnetic Feshbach resonance,

$$K^{c0}_l(\epsilon,B) = K^{c0}_{{\rm bg}l}\left[1 + \frac{d_{El}}{\epsilon - \delta\mu_l(B - B_{0l}) - d_{El}}\right]. \tag{18}$$

It is a parametrization in terms of four parameters B_{0l}, $K^{c0}_{{\rm bg}l}$, $\delta\mu_l$, and d_{El}, with the condition of $K^{c0}_{{\rm bg}l}d_{El} < 0$ due to $\Gamma^{c0}_l > 0$. These parameters, together with either the C_6 coefficient or the corresponding energy scale s_E for a total of five parameters, provide a complete characterization of atomic interaction around a magnetic Feshbach resonance, through Eqs. (2) and (3). It is applicable for all partial waves l and for either broad or narrow Feshbach resonances (the precise definition of which will be addressed in in Sec. III C) or anything in between. It fails only in the special case of having a *background* bound or quasibound state right at the threshold, which can happen only by pure coincidence. This special case, together with an alternative parametrization of magnetic Feshbach resonances that is applicable for all cases, is discussed in Appendix B.

B. Tuning of the scattering lengths and generalized scattering lengths

Contained in the parametrization of K_l^{c0} is the magnetic-field dependence of the scattering length or the generalized scattering length for an abitrary l. Defining $K_l^{c0}(B) \equiv K_l^{c0}(\epsilon = 0, B)$ to simplify the notation, we have from Eq. (18)

$$K_l^{c0}(B) = K_{\text{bg}l}^{c0}\left(1 - \frac{d_{Bl}}{B - B_{0l} + d_{Bl}}\right), \tag{19}$$

where $d_{Bl} = d_{El}/\delta\mu_l$. Upon substitution into Eq. (6), we obtain, for $\widetilde{a}_{\text{bg}l} \neq 0$,

$$\widetilde{a}_l(B) = \bar{a}_l\left[(-1)^l + \frac{1}{K_l^{c0}(B)}\right] \tag{20}$$

$$= \widetilde{a}_{\text{bg}l}\left(1 - \frac{\Delta_{Bl}}{B - B_{0l}}\right). \tag{21}$$

Here, $\widetilde{a}_{\text{bg}l}$ is the (generalized) background scattering length defined earlier by Eq. (11), and

$$\Delta_{Bl} = -d_{Bl}/\left[1 + (-1)^l K_{bgl}^{c0}\right], \tag{22}$$

$$= -\left[1 - (-1)^l \frac{1}{\widetilde{a}_{\text{bg}l}/\bar{a}_l}\right] d_{Bl}. \tag{23}$$

For $\widetilde{a}_{\text{bg}l} = 0$, we obtain

$$\widetilde{a}_l(B) = -(-1)^l \bar{a}_l \frac{d_{Bl}}{B - B_{0l}}. \tag{24}$$

Equation (21) shows that around a magnetic Feshbach resonance of arbitrary l, the (generalized) scattering length is tuned in a similar fashion by the magnetic field as around an s wave resonance and can be parametrized in a similar manner [12,13,25].

The parametrization of the s wave scattering length in the form of Eq. (21) has been popular for a good reason: every parameter in it has the simplest and the most direct experimental interpretation. It is worth pointing out, however, that theoretically it is not the most general parametrization possible as it fails for both $\widetilde{a}_{\text{bg}l} = \infty$ and $\widetilde{a}_{\text{bg}l} = 0$. As discussed in Appendix B, the failure of Eq. (21) at $\widetilde{a}_{\text{bg}l} = \infty$, and the corresponding failure of Eqs. (18) and (19) at $K_{\text{bg}l}^{c0} = 0$, is a necessary sacrifice for using B_{0l}, which has a more direct physical interpretation than the $\bar{B}_{0l}$ parameter of Appendix B but does not exist for $\widetilde{a}_{\text{bg}l} = \infty$. Its failure at $\widetilde{a}_{\text{bg}l} = 0$ is the price we pay for using the parameter Δ_{Bl}. An alternative parametrization of the scattering length, which remains applicable for $\widetilde{a}_{\text{bg}l} = 0$, is

$$\widetilde{a}_l(B) = \widetilde{a}_{\text{bg}l} + \frac{\widetilde{a}_{\text{bg}l} - (-1)^l \bar{a}_l}{(B - B_{0l})/d_{Bl}}. \tag{25}$$

It can be obtained, e.g., by substituting Eq. (23) for Δ_{Bl} into Eq. (21). This parametrization is well defined and reduces to Eq. (24) for $\widetilde{a}_{\text{bg}l} = 0$.

C. Parametrization in terms of scaled parameters and the definitions of "broad" and "narrow" resonances

Of the five parameters required to completely characterize the atomic interaction around a Feshbach resonance, such as B_{0l}, $K_{\text{bg}l}^{c0}$, d_{El}, $\delta\mu_l$, and s_E (or C_6), two of them, $K_{\text{bg}l}^{c0}$ and d_{El}, can be replaced by $\widetilde{a}_{\text{bg}l}$ and Δ_{Bl}, used in the parametrization of the (generalized) scattering length. The resulting parametrization, in terms of B_{0l}, $\widetilde{a}_{\text{bg}l}$, Δ_{Bl}, $\delta\mu_l$, and s_E (or C_6), gives an alternative that is the most direct generalization of the s wave parametrization [13] to other partial waves. Both sets, however, have the limitation that they are not fully transparent to the distinction between broad and narrow resonances.

The effective single channel K_l^{c0} parameter for a magnetic Feshbach resonance, as characterized, e.g., by Eq. (18), is generally energy-dependent. Depending on the relative importance of this energy variation, as compared to those due to the long-range van der Waals interaction, a Feshbach resonance can be classified either as "broad" or "narrow." For a broad Feshbach resonance, the energy dependence of K_l^{c0} is insignificant compared to those induced by the van der Waals interaction. The atomic interaction around such a resonance follows, to a large extent, the single-channel universal behavior of paper I [7] with a tunable (generalized) scattering length. A narrow Feshbach resonance corresponds to the opposite limit in which the energy dependence of K_l^{c0} dominates. The atomic interaction around such a resonance can differ completely from the single-channel universal behavior.

To better characterize the relative importance of the energy dependence of K_l^{c0} and therefore the definition of broad and narrow resonances, we need to first put it on the same energy scale as the other energy-dependent functions, namely on the energy scale $s_E = (\hbar^2/2\mu)(1/\beta_6)^2$ that is associated with the van der Waals interaction [7]. Defining

$$g_{\text{res}} = d_{El}/s_E \tag{26}$$

and

$$B_s = (B - B_{0l})/d_{Bl}, \tag{27}$$

Eq. (18) can be written as

$$K_l^{c0}(\epsilon_s, B_s) = K_{\text{bg}l}^{c0}\left[1 + \frac{g_{\text{res}}}{\epsilon_s - g_{\text{res}}(B_s + 1)}\right]. \tag{28}$$

It describes K_l^{c0} as a function of the scaled energy ϵ_s and a scaled magnetic field B_s using two dimensionless parameters, $K_{\text{bg}l}^{c0}$ and g_{res}, the meaning of which are illustrated in Fig. 1. $K_{\text{bg}l}^{c0}$ is the background K_l^{c0}, namely its value away from the resonance. g_{res} is a measure of the width of the resonance. More specifically, $K_l^{c0}(\epsilon_s, B)$ goes to infinity at $\epsilon_s = g_{\text{res}}(B_s + 1)$. It crosses zero $\epsilon_s = g_{\text{res}} B_s$. The distance between the two locations is $|g_{\text{res}}|$, which measures the width of resonance.

The parametrization of K_l^{c0} using Eq. (28) divides the parameters characterizing a magnetic Feshbach resonance into three parameters, B_{0l}, d_{Bl}, and s_E, for location and scaling, and two dimensionless parameters, $K_{\text{bg}l}^{c0}$ and g_{res}, for the shape. Feshbach resonances with the same shape parameters differ from each other only in scaling. The condition of $\Gamma_l^{c0} > 0$ implies that the two shape parameters, $K_{\text{bg}l}^{c0}$ and g_{res}, are constrained by $K_{\text{bg}l}^{c0} g_{\text{res}} < 0$.

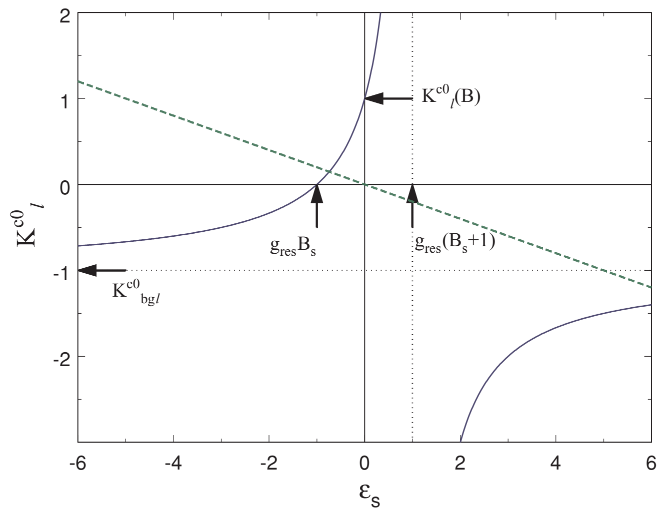

FIG. 1. (Color online) Illustrations of the parameters describing the energy dependence of $K_l^{c0}(\epsilon_s,B_s)$ on the van der Waals energy scale. $\Gamma_l^{c0}>0$ ($K_{\text{bg}l}^{c0}g_{\text{res}}<0$) implies that K_l^{c0} is piecewise monotonically increasing function of energy. The significance of its energy dependence is determined by comparing it with that induced by the van der Waals interaction, the order-of-magnitude of which can be measured by the energy dependence of the $\theta_l\approx-\epsilon_s/(2l+3)(2l-1)$ function. The dashed line illustrates the θ_l function for $l=1$. The energy variation of θ_l is less significant for higher partial waves.

With the parametrization given by Eq. (28), we are now in position for rigorous definitions of broad and narrow resonances. The g_{res} parameter, which measures the width of the resonance on the scale of s_E, gives a rough, yet still imprecise, classification of "broad" ($|g_{\text{res}}|\gg 1$) and "narrow" ($|g_{\text{res}}|\ll 1$). It is not precise because the energy variation due to the van der Waals interaction over a scale of s_E is different for different partial waves. This is especially true for large l, for which the energy variation around the threshold due to the van der Waals interaction is much less significant than that for the s wave.

For a more precise definition of "broad" and "narrow," we first recognize that the leading energy variation due to the van der Waals interaction is characterized by the $\theta_l\approx-\epsilon_s/(2l+3)(2l-1)$ function defined by Eq. (34) of paper I [7] [repeated as Eq. (43) in Sec. IV]. This energy variation, as measured by $|\partial\theta_l/\partial\epsilon_s(\epsilon_s=0)|=|1/(2l+3)(2l-1)|$, is what should be compared with the energy variation of the K_l^{c0} at zero energy, as measured by $\partial K_l^{c0}/\partial\epsilon_s(\epsilon_s=0,B_s=0)=-K_{\text{bg}l}^{c0}/g_{\text{res}}$. This leads to the definition of an auxiliary parameter

$$\zeta_{\text{res}}\equiv\frac{g_{\text{res}}}{(2l+3)(2l-1)K_{\text{bg}l}^{c0}}\ , \tag{29}$$

which gives a precise characterization of "broad" and "narrow." For a broad resonance with $|\zeta_{\text{res}}|\gg 1$, the energy variation of the effective short-range parameter is insignificant compared to that due to the van der Waals interaction, just like the case of a single channel [7]. The atomic interaction around such a resonance can be expected to follow the single-channel universal behavior. For a narrow resonance with $|\zeta_{\text{res}}|\ll 1$, the energy variation of the effective short-range parameter dominates. Within such a resonance, the atomic interaction deviates substantially from the single-channel behavior. The constraint $K_{\text{bg}l}^{c0}g_{\text{res}}<0$ implies that ζ_{res} is always positive for $l=0$ and always negative for $l\geqslant 1$. Specializing to the s wave, the ζ_{res} parameter is similar in spirit to the parameter s_{res} of Chin *et al.* [13], which is equivalent to the $1/\eta$ parameter of Köhler *et al.* [12].

To finish our discussion on parametrization, we summarize here the explicit relations between two sets of parameters that we will use for the complete characterization of a Feshbach resonance. The first set is B_{0l}, $\widetilde{a}_{\text{bg}l}$, Δ_{Bl}, $\delta\mu_l$, and s_E (or C_6), which is more closely correlated with the parametrization of the (generalized) scattering length and the standard s wave parametrization [13]. The second set is B_{0l}, $K_{\text{bg}l}^{c0}$, g_{res}, d_{Bl}, and s_E (or C_6), which is much more convenient with the QDT expansion of Sec. IV and correlates much more closely with the distinction of broad and narrow resonances. They differ in three parameters that are related by

$$K_{\text{bg}l}^{c0}=\frac{1}{\widetilde{a}_{\text{bg}l}/\bar{a}_l-(-1)^l}\ , \tag{30}$$

$$g_{\text{res}}=-\frac{\widetilde{a}_{\text{bg}l}/\bar{a}_l}{\widetilde{a}_{\text{bg}l}/\bar{a}_l-(-1)^l}\left(\frac{\delta\mu_l\Delta_{Bl}}{s_E}\right)\ , \tag{31}$$

$$d_{Bl}=-\frac{\widetilde{a}_{\text{bg}l}/\bar{a}_l}{\widetilde{a}_{\text{bg}l}/\bar{a}_l-(-1)^l}\Delta_{Bl}\ . \tag{32}$$

The condition of $\Gamma_l^c>0$ translates into the constraint $\delta\mu_l\widetilde{a}_{\text{bg}l}\Delta_{Bl}>0$ for the first set of parameters and into $K_{\text{bg}l}^{c0}g_{\text{res}}<0$ for the second. Table I gives examples of both sets of parameters for selective s wave magnetic Feshbach resonances. The first set is taken from Table IV of Chin *et al.* [13]. The second set is calculated from the first using Eqs. (30)–(32). They are given here both for convenient applications of the QDT expansion and to illustrate the vast range of ζ_{res}, from very narrow $|\zeta_{\text{res}}|\ll 1$ to very broad $|\zeta_{\text{res}}|\gg 1$. Since the parametrization is new for nonzero partial waves, no parameters are yet available for them. Tentative theoretical predictions of resonances and their parameters for nonzero partial waves, using MQDT as briefly outlined in Sec. II, will be presented elsewhere. It is hoped that they will stimulate further experiment and theory for their more precise characterization. Previous works on nonzero partial waves, such as those in Refs. [13,31,40,48–55], can also be reanalyzed to extract the parameters.

The second set of parameters describes K_l^{c0} through Eq. (28). A useful variation, which relates K_l^{c0} explicitly to its value at zero energy, is given by

$$K_l^{c0}(\epsilon_s,B_s)=\frac{K_l^{c0}(B_s)-K_{\text{bg}l}^{c0}\eta(B_s)\epsilon_s}{1-\eta(B_s)\epsilon_s}, \tag{33}$$

where

$$K_l^{c0}(B_s)=K_l^{c0}(\epsilon_s=0,B_s)=K_{\text{bg}l}^{c0}\frac{B_s}{B_s+1} \tag{34}$$

is the value of K_l^{c0} at zero energy, given earlier by Eq. (19), expressed in terms of the scaled magnetic field, and we have defined

$$\eta(B_s)=\frac{1}{g_{\text{res}}(B_s+1)}. \tag{35}$$

This representation of K_l^{c0} makes it clear that $K_l^{c0}(\epsilon_s,B_s)\sim K_l^{c0}(B_s)$ in the broad-resonance limit of $|g_{\text{res}}|\to\infty$. It also

TABLE I. Sample parameters for selective s wave magnetic Feshbach resonances, illustrating a vast range of $\zeta_{\rm res}$ values, from very narrow ($|\zeta_{\rm res}| \ll 1$) to very broad ($|\zeta_{\rm res}| \gg 1$). It shows, for example, that ^{6}Li-^{6}Li and ^{133}Cs-^{133}Cs systems have the best resonances for the purpose of investigating universal behaviors. Here, a_0 is the Bohr radius and μ_B is the Bohr magneton. The data sets of B_{0l}, Δ_{Bl}, $a_{{\rm bg}l}$, and $\delta\mu_l$ are taken from Table IV of Chin *et al.* [13]. The channel identification also follows the same reference. Note that there are many resonances that are not broad. The atomic interaction around them does not follow single-channel universal behavior and is much better described using the QDT expansion presented here.

System	$s_E/k_B(\mu{\rm K})$	Ch.	$B_{0l}({\rm G})$	$\Delta_{Bl}({\rm G})$	$a_{{\rm bg}l}/a_0$	$\delta\mu_l/\mu_B$	$K^{c0}_{{\rm bg}l}$	$g_{\rm res}$	$d_{Bl}({\rm G})$	$\zeta_{\rm res}$	References
^{6}Li^{6}Li	7368	*ab*	834.1	−300	−1405	2.0	−0.02083	5.356	293.8	85.73	[33]
		ac	690.4	−122.3	−1727	2.0	−0.01701	2.192	120.2	42.96	[33]
		bc	811.2	−222.3	−1490	2.0	−0.01966	3.974	217.9	67.37	[33]
		ab	543.25	0.1	60	2.0	0.9923	−0.00363	−0.1992	0.00122	[34]
^{7}Li^{7}Li	5849	*aa*	736.8	−192.3	−25	1.93	−0.5540	1.901	85.76	1.144	[13,35,36]
^{23}Na^{23}Na	933.1	*cc*	1195	−1.4	62	−0.15	2.255	−0.04921	4.557	0.00727	[13,37,38]
		aa	907	1	63	3.8	2.143	−0.8597	−3.143	0.1337	[13,37,38]
		aa	853	0.0025	63	3.8	2.143	−0.00215	−0.00786	0.00033	[13,37,38]
^{40}K^{40}K	257.3	*ab*	202.1	8.0	174	1.68	0.5542	−5.454	−12.43	3.280	[13,39]
		ac	224.2	9.7	174	1.68	0.5542	−6.613	−15.07	3.978	[13,40]
^{85}Rb^{85}Rb	75.58	*ee*	155.04	10.7	−443	−2.33	−0.1506	18.82	−9.089	41.66	[41]
^{87}Rb^{87}Rb	72.99	*aa*	1007.4	0.21	100	2.79	3.759	−2.566	−0.9994	0.2275	[13,42,43]
		aa	911.7	0.0013	100	2.71	3.759	−0.01543	−0.00619	0.00137	[13,44]
		aa	685.4	0.006	100	1.34	3.759	−0.03521	−0.02855	0.00312	[13,43,44]
		aa	406.2	0.0004	100	2.01	3.759	−0.003521	−0.00190	0.00031	[13,44]
		ae	9.13	0.015	99.8	2.00	3.795	−0.1324	−0.07193	0.01163	[45]
^{133}Cs^{133}Cs	31.97	*aa*	−11.7	28.7	1720	2.30	0.05945	−146.9	−30.41	823.9	[13,46,47]
		aa	547	7.5	2500	1.79	0.04015	−29.34	−7.801	243.5	[13]
		aa	800	87.5	1940	1.75	0.05235	−338.6	−92.08	2156	[13]

makes it easier, if ever desirable, to represent $K^{c0}_l(\epsilon_s,B_s)$ in terms of (generalized) scattering length and (generalized) background scattering length, using Eqs. (30)–(32), and

$$K^{c0}_l(B_s) = \frac{1}{\widetilde{a}_l(B)/\bar{a}_l - (-1)^l}\ , \tag{36}$$

which is a direct consequence of Eq. (20).

IV. QDT EXPANSION FOR ULTRACOLD SCATTERING AROUND A MAGNETIC FESHBACH RESONANCE

In deriving the QDT expansion for single-channel ultracold scattering of paper I [7], the only quantities expanded are the universal QDT functions, with no assumptions made about the behavior of the short-range parameter K^{c0}_l, including its energy dependence. Thus, the same expansion is applicable to the effective single-channel problem that describes the magnetic Feshbach resonance. Specifically, we have for $\epsilon_s > 0$ [7],

$$\tan\delta_l \approx K^{(B)}_l(\epsilon_s) + K^{(D)}_l(\epsilon_s,B_s), \tag{37}$$

where

$$\begin{aligned} K^{(B)}_l &\approx -\pi(\nu-\nu_0) \\ &\approx \frac{3\pi}{(2l+5)(2l+3)(2l+1)(2l-1)(2l-3)}\epsilon_s^2 \end{aligned} \tag{38}$$

is the Born term (see, e.g., Ref. [56]), and

$$K^{(D)}_l(\epsilon_s,B_s) = -\widetilde{A}_{sl}(\epsilon_s,B_s)k_s^{2l+1} \tag{39}$$

describes the deviation from the Born term. Here,

$$\begin{aligned} &\widetilde{A}_{sl}\ (\epsilon_s,B_s) \\ &= \bar{a}_{sl}\left[(-1)^l + \frac{1+K^{c0}_l(\epsilon_s,B_s)\theta_l}{K^{c0}_l(\epsilon_s,B_s)-\theta_l-\pi(\nu-\nu_0)/2}\right], \end{aligned} \tag{40}$$

$$= \bar{a}_{sl}\left[(-1)^l + \frac{(2l+3)(2l-1)-K^{c0}_l(\epsilon_s,B_s)\epsilon_s}{(2l+3)(2l-1)K^{c0}_l(\epsilon_s,B_s)+\epsilon_s+w_l\epsilon_s^2}\right], \tag{41}$$

with the w_l in Eq. (41) being given by

$$w_l = \frac{3\pi}{2(2l+5)(2l+1)(2l-3)} \tag{42}$$

and the θ_l in Eq. (40) being given by

$$\theta_l \approx -\frac{1}{(2l+3)(2l-1)}\epsilon_s. \tag{43}$$

Equations (37)–(41), with a K^{c0}_l that depends explicitly on energy and parametrically on the magnetic field B, as described by either Eq. (28), Eq. (33), or Eq. (B1) of Appendix B, give the QDT expansion for scattering around a magnetic Feshbach resonance in an arbitrary partial wave l. It is applicable to both broad and narrow resonances, or anything in between, and has the same energy range of applicability as its single-channel counterpart, limited only by ϵ_s being much less than the critical scale energy $\epsilon_{\rm scl}$, as discussed in more detail in paper I [7]. There is no restriction on the magnetic field except that imposed by the validity of the isolated resonance.

While the QDT expansion for scattering around a magnetic Feshbach resonance may be formally similar to QDT expansion for true single-channel cases [7], it contains considerable new physics beyond those of a single channel, including dramatically different behaviors for broad and narrow resonances. For broad resonances with $|g_{\rm res}| \gg 1$, or more precisely $|\zeta_{\rm res}| \gg 1$, the energy dependence of $K_l^{c0}(\epsilon_s, B_s)$ is negligible. The QDT expansion approaches the single-channel universal behavior [7] defined by replacing $K_l^{c0}(\epsilon_s, B_s)$ in Eqs. (40) and (41) with its zero-energy value of $K_l^{c0}(B_s)$. In such cases, multichannel scattering behaves the same as a single-channel, with a tunable (generalized) scattering length not only at the threshold but over a range of energies around the threshold with an energy dependence determined primary by the van der Waals interaction. This is illustrated in Figs. 2 and 3 using a broad resonance of ^{6}Li-^{6}Li. Narrow resonances behave very differently with a much more complex energy dependence that is determined both by the properties of the resonance and by the van der Waals interaction. They change scattering in a narrow range of energies around the resonance. Away from it, atomic interaction evolves toward a single-channel universal behavior determined not by $K_l^{c0}(B_s)$, but by $K_{{\rm bg}l}^{c0}$ or equivalently the background (generalized) scattering length $\widetilde{a}_{{\rm bg}l}$, with

$$K_l^{(D)} \sim -\bar{a}_{sl} k_s^{2l+1} \times \left[(-1)^l + \frac{(2l+3)(2l-1) - K_{{\rm bg}l}^{c0}\epsilon_s}{(2l+3)(2l-1)K_{{\rm bg}l}^{c0} + \epsilon_s + w_l \epsilon_s^2} \right]. \quad (44)$$

Figures 2 and 3 contain illustrations of narrow-resonance behavior using an example from ^{6}Li-^{6}Li scattering. Further conceptual understanding of the differences between broad and narrow resonances can be found in the next section, in connections with the generalized effective range expansion and examples for infinite and zero (generalized) scattering lengths.

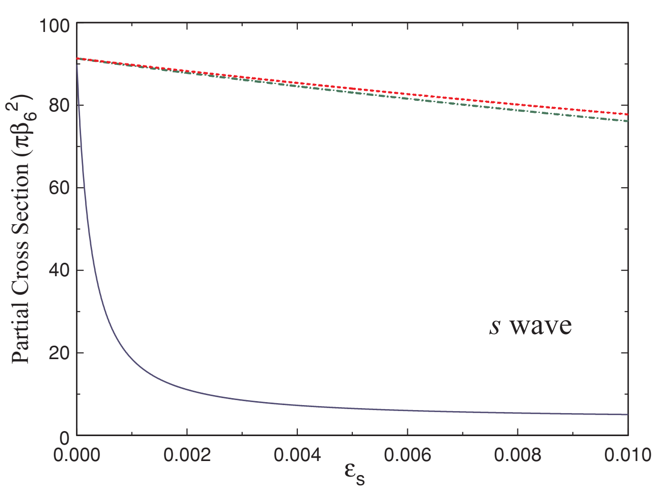

FIG. 2. (Color online) Comparison of near-threshold s wave scattering properties for narrow and broad Feshbach resonances. The solid line represents results for the ^{6}Li ab channel narrow resonance located at 543 G. The dash-dot line represents results for the ^{6}Li ab channel broad resonance located at 834 G. In both cases, magnetic fields are chosen to give the same s wave scattering length corresponding to $a_{l=0}(B)/\bar{a}_{l=0} = +10$. The figure also shows that the Feshbach resonance at 834 G is sufficiently broad that the scattering properties around it are well approximated by the single-channel universal behavior (dashed line).

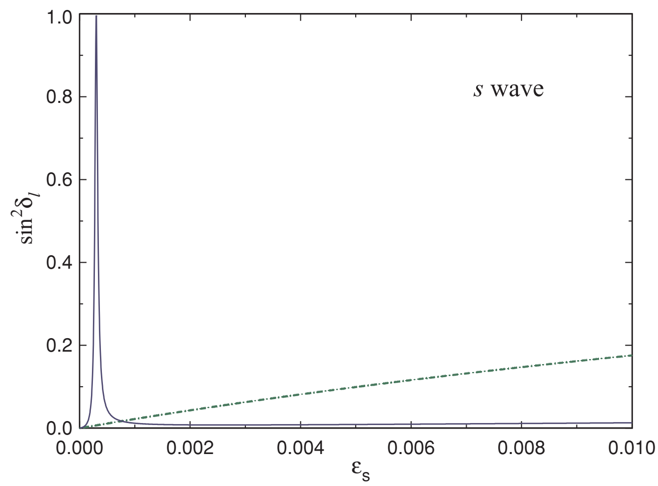

FIG. 3. (Color online) The same as described in the legend of Fig. 2 except it is for $a_{l=0}(B)/\bar{a}_{l=0} = -10$. $\sin^2\delta_l$ is plotted, instead of the partial cross section to give better visibility to both sets of data on the same figure. The single-channel universal behavior is indistinguishable from the broad-resonance results (dash-dot line) and is not plotted. Note that even though both set of data correspond to the same scattering length, the case of narrow Feshbach (solid line) has a resonance feature in the threshold region that is absent for a broad Feshbach. (See also Ref. [13].)

As an illustration of the breadth of the physics contained in the QDT expansion for a magnetic Feshbach resonance, Fig. 4 shows its description of an avoided crossing between a narrow p-wave Feshbach-shape resonance and a background p-wave shape resonance in the threshold region. It is an example of the coupling of a bound state to a highly "structured" continuum and is used here to emphasize that the *only* restriction on the applicability of the QDT expansion is $\epsilon_s \ll \epsilon_{\rm scl}$ [7].

For nonzero partial waves, a Feshbach resonance above the threshold manifests itself as a Feshbach-shape resonance. The qualitative characteristics of such a resonance, such as its

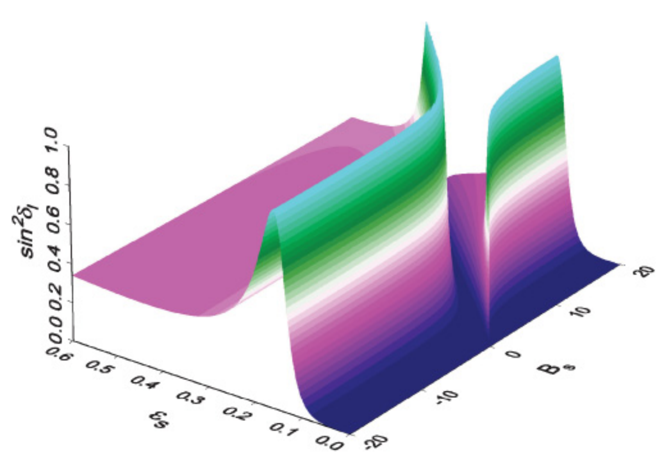

FIG. 4. (Color online) A plot of $\sin^2\delta_l$ vs ϵ_s and B_s for the p wave, with parameters $K_{{\rm bg}l}^{c0} = -0.03$ and $g_{\rm res} = 0.02$. It illustrates the avoided crossing between a background shape resonance located in the threshold region and a narrow Feshbach-shape resonance.

position and width, are contained within the QDT expansion, in a manner similar to the case of a single channel [7]. We defer their discussions to a following paper, since they need to be combined with the binding energy of a Feshbach molecule to give a complete picture of the evolution of a resonance across the threshold.

V. SAMPLE APPLICATIONS

A. The generalized effective range expansion around a magnetic Feshbach resonance

One of the ways to understand some of the physics contained in the QDT expansion for atomic interaction around a magnetic Feshbach resonance is through the generalized effective range expansion contained within it. As in paper I [7], the QDT expansion, given by Eqs. (37)–(41), can be *approximated* by a generalized effective range expansion:

$$k^{2l+1}\cot\left(\delta_l-\delta_l^{(B)}\right)=-\frac{1}{\widetilde{a}_l}+\frac{1}{2}\widetilde{r}_{el}k^2+O(k^4\ln k),\quad(45)$$

where $\delta_l^{(B)}=-\pi(\nu-\nu_0)$ is approximated by Eq. (38). It reduces to the standard effective range expansion [2–4] for $l=0$, and serves to define the generalized scattering length and the generalized effective range for other l [7]. For scattering around a magnetic Feshbach resonance, both the (generalized) scattering length and the (generalized) effective range become magnetic-field dependent. The (generalized) scattering length, $\widetilde{a}_l$, is tuned by the magnetic field according to Eq. (21). From the QDT expansion, it is straightforward to show that the (generalized) effective range is given by

$$\begin{aligned}\widetilde{r}_{el}(B)&=-\frac{2\bar{a}_l\beta_6^2}{(2l+3)(2l-1)[\widetilde{a}_l(B)]^2}\left\{1+\left[(-1)^l-\frac{\widetilde{a}_l(B)}{\bar{a}_l}\right]^2\right\}\\&\quad-\left(\frac{\hbar^2}{\mu\widetilde{a}_{bgl}\delta\mu_l\Delta_{Bl}}\right)\left(\frac{\Delta_{Bl}}{B-B_{0l}-\Delta_{Bl}}\right)^2,\quad(46)\\&=-\frac{2\bar{a}_l\beta_6^2}{(2l+3)(2l-1)[\widetilde{a}_l(B)]^2}\left\{1+\left[(-1)^l-\frac{\widetilde{a}_l(B)}{\bar{a}_l}\right]^2\right\}\\&\quad+\left(\frac{1}{\zeta_{res}}\right)\frac{2\beta_6^2}{(2l+3)(2l-1)\bar{a}_l}\left(\frac{\Delta_{Bl}}{B-B_{0l}-\Delta_{Bl}}\right)^2,\quad(47)\\&=-\frac{2\bar{a}_l\beta_6^2}{(2l+3)(2l-1)[\widetilde{a}_l(B)]^2}\left\{1+\left[(-1)^l-\frac{\widetilde{a}_l(B)}{\bar{a}_l}\right]^2\right\}\\&\quad+\left(\frac{1}{\zeta_{res}}\right)\frac{2\bar{a}_l\beta_6^2}{(2l+3)(2l-1)[\widetilde{a}_l(B)]^2}\left[\frac{\widetilde{a}_l(B)-\widetilde{a}_{bgl}}{\bar{a}_l}\right]^2.\quad(48)\end{aligned}$$

It consists of two terms. The first corresponds to the single-channel universal behavior of paper I [7] and is due to the long-range van der Waals interaction. The second term is due to the energy dependence of the effective K_l^{c0} that comes from the coupling to a bound state in closed channels. It is given here in three different forms with distinctive insights.

In the broad-resonance limit of $|\zeta_{res}|\to\infty$, the energy dependence of the effective K_l^{c0} is negligible, and the result reduces to the single-channel universal behavior [7], in which the (generalized) effective range is uniquely determined by the (generalized) scattering length, $\widetilde{a}_l(B)$, independent of the other details of the resonance. Within a narrow resonance, the (generalized) effective range is changed substantially from the universal behavior, which is one way to understand its substantially different near-threshold behavior as illustrated in Figs. 2 and 3. This change, as characterized by the second terms in Eqs. (46)–(48), depends sensitively both on the location within the resonance and on specific characteristics of the resonance, which are described, for instance, by the ζ_{res} and $\widetilde{a}_{bgl}$ parameters in Eq. (48). Away from the resonance, namely for $|B-B_{0l}|\gg|\Delta_{Bl}|$, the second term goes away and the (generalized) effective range evolves back towards a single-channel universal result [7] determined by the (generalized) background scattering length, $\widetilde{a}_{bgl}$.

It is useful to note that the contribution to the (generalized) effective range due to the coupling to a bound state in closed channels, the second term, is always negative. It comes from the constraint of $\Gamma^c>0$ discussed earlier, which translates into $\delta\mu_l\widetilde{a}_{bgl}\Delta_{Bl}>0$ for parameters used in Eq. (46), and into $\zeta_{res}>0$ for $l=0$ and $\zeta_{res}<0$ for all other partial waves, for parameters used in Eqs. (47) and (48) (see Sec. III C). The implication is that $\widetilde{r}_{el}(B)$ is always negative for $l>0$ as its corresponding first term is also always negative [7]. For the s wave, the two terms are always of opposite signs and the end result can be either positive or negative.

Specializing to the case of s wave, the second term in Eq. (46) is the expression for the effective range that is adopted by Zinner and Thogersen [23] in their investigation of Bose-Einstein condensate (BEC) around a narrow Feshbach resonance. It comes from the work of Bruun *et al.* [57], which ignores the effect of van der Waals interaction. Having only the second term in their theories means that their results are applicable only for narrow resonances ($|\zeta_{res}|\ll 1$) and only in the resonance region where the scattering length differs substantially from the background scattering length.

We note that while the generalized effective range expansion is useful both as a connection to previous theories and for studies of dilute quantum gases at ultracold temperatures [20,22,23], it has its limitations. Similar to the case of a single channel [7], it fails around $\widetilde{a}_l(B)=0$, and has generally a much more limited range of applicability compared to the QDT expansion, from which it is derived.

B. Examples of infinite and zero (generalized) scattering lengths

Both for the purpose of illustrating explicit energy dependences contained in the QDT expansion for magnetic Feshbach resonances and to facilitate future applications, we give here explicit QDT expansion for two special cases of interest in cold-atom physics. One is the case of infinite (generalized) scattering length, the so-called unitary limit. The other is the case of zero (generalized) scattering length.

For infinite (generalized) scattering length, which occurs at $B=B_{0l}$, or equivalently, at $B_s=0$, the QDT expansion for

$K_l^{(D)}$, Eqs. (39) and (41), becomes

$$K_l^{(D)} = \bar{a}_{sl} k_s^{2l-1} \frac{(2l+3)(2l-1)g_{\rm res} - \epsilon_s[(2l+3)(2l-1) - K^{c0}_{{\rm bg}l}\epsilon_s]}{(2l+3)(2l-1)K^{c0}_{{\rm bg}l} + \epsilon_s + w_l\epsilon_s^2 - g_{\rm res}(1+w_l\epsilon_s)} - (-1)^l \bar{a}_{sl} k_s^{2l+1}. \quad (49)$$

In the broad resonance limit of $|g_{\rm res}| \to \infty$, it reduces to Eq. (51) of paper I [7]. For a narrow resonance with $|\zeta_{\rm res}| \ll 1$, scattering around the threshold follows that of an infinite (generalized) scattering length, with $K_l^{(D)} \sim k_s^{2l-1}$ [7], only in a very small energy range of $0 < \epsilon_s \ll |g_{\rm res}|$ around the threshold. Outside this region, namely for energies $\epsilon_s \gg |g_{\rm res}|$, it evolves into the background scattering described by Eq. (44), as discussed in the previous section in a more general context.

Another special case of interest, where the effective range expansion, including the generalized version of Sec. V A, fails completely, is that of zero (generalized) scattering length. It occurs at the magnetic field $B = B_{0l} + \Delta_{Bl}$ or, equivalently, at $B_s = -[1 + (-1)^l K^{c0}_{{\rm bg}l}]^{-1}$, where Eqs. (39) and (41) becomes

$$K_l^{(D)} = \bar{a}_{sl} k_s^{2l+3} \times \frac{K^{c0}_{{\rm bg}l} g_{\rm res}(2+w_l\epsilon_s) - [(-1)^l + K^{c0}_{{\rm bg}l}]\{(2l+3)(2l-1)K^{c0}_{{\rm bg}l} + \epsilon_s + w_l\epsilon_s^2 + (-1)^l[(2l+3)(2l-1) - K^{c0}_{{\rm bg}l}\epsilon_s]\}}{K^{c0}_{{\rm bg}l} g_{\rm res}[(2l+3)(2l-1)K^{c0}_{{\rm bg}l} - (-1)^l\epsilon_s - (-1)^l w_l\epsilon_s^2] + (-1)^l[(-1)^l + K^{c0}_{{\rm bg}l}]\epsilon_s[(2l+3)(2l-1)K^{c0}_{{\rm bg}l} + \epsilon_s + w_l\epsilon_s^2]}, \quad (50)$$

In the broad resonance limit of $|g_{\rm res}| \to \infty$, it reduces to Eq. (49) of paper I. For a narrow resonance with $|\zeta_{\rm res}| \ll 1$, it behaves as scattering of zero (generalized) scattering length, with $K_l^{(D)} \sim k_s^{2l+3}$ [7], only in a small energy range of $0 < \epsilon_s \ll |g_{\rm res}|$. Outside this range, it becomes that determined by the background scattering, as given by Eq. (44). Equations (49) and (50) illustrate the kind of energy dependences contained in the QDT expansion for magnetic Feshbach resonances, and the complexity required to describe two types of behaviors, one determined by $K_l^{c0}(B_s)$ or $\widetilde{a}_l(B)$ sufficiently close to the threshold and one by $K^{c0}_{{\rm bg}l}$ or $\widetilde{a}_{{\rm bg}l}$ away from the resonance, and the evolution between the two.

VI. CONCLUSIONS

In conclusion, we have presented an analytic description of a magnetic Feshbach resonance in an arbitrary partial wave l and the atomic scattering around it at ultracold temperatures. It is derived by showing, in a very general context, that a multichannel problem below the second threshold, all the way through the bound spectrum, is equivalent to an effective single-channel problem with a generally energy- and partial-wave-dependent short-range parameter. The relative significance of this energy dependence, in comparison with those induced by the long-range interaction, leads to the classification of Feshbach resonances of arbitrary l into broad and narrow resonances, with vastly different scattering characteristics around the threshold.

We have shown that, except for the special case of $K^{c0}_{{\rm bg}l} = 0$ (corresponding to $\widetilde{a}_{{\rm bg}l} = \infty$, and discussed further in Appendix B), a magnetic Feshbach resonance of arbitrary l can be parametrized in a similar fashion as an s wave Feshbach resonance [12,13,25], in terms of five parameters, which can be either B_{0l}, $\widetilde{a}_{{\rm bg}l}$, Δ_{Bl}, $\delta\mu_l$, and s_E (or C_6) or B_{0l}, $K^{c0}_{{\rm bg}l}$, $g_{\rm res}$, d_{Bl}, and s_E (or C_6). These parameters, together with the QDT expansion, give accurate analytic descriptions of atomic interactions around a magnetic Feshbach resonance, not only of the scattering properties presented here but also of the binding energies of a Feshbach molecule and of scattering at negative energies [10], to be presented in a following publication. Such descriptions can now be incorporated into theories of atomic interaction in an optical lattice [16], using, e.g., the multiscale QDT of Ref. [17], and theories of few-atom and many-atom systems around a Feshbach resonance, especially around a resonance that is not broad (See, e.g., Refs. [14,19–23]).

Accurate determinations of Feshbach parameters will generally require a combination of theoretical and experimental efforts, as have been done previously for the s wave [13]. The derivation of the parametrization, as presented in Secs. II and III and Appendix B, also constitutes an outline of a theory for these parameters and how they can be computed from the MQDT formulation for atomic interaction in a magnetic field [24,31]. The detailed implementation of the theory and results for specific systems will be presented elsewhere. They will answer questions such as "Are there any broad resonances in p or higher partial waves, and which systems have them?" By comparing theoretical predications and experimental measurements over a wide range of magnetic fields, we will also learn about the energy and the partial-wave dependences of the short-range K_S^c and K_T^c parameters of Ref. [24], which are due to interactions of shorter range, such as the C_8/r^8 term in the potential. Such variations, once determined, will allow MQDT to provide a basically analytic description of atomic interactions over a wide range of temperatures, from 0 K to 1 K, and beyond [24].

Finally, it should be clear that while the focus of this article is on magnetic Feshbach resonances, many of the concepts are much more generally applicable. In particular, the theoretical development followed here provides a very general methodology on how analytic descriptions of certain aspects of a multichannel problem may be developed and understood. The theory is also a necessary step toward resolving one remaining difficulty in the analytic description of ultracold atomic interaction, which is to efficiently incorporate the weak magnetic dipole-dipole and second-order spin-orbit interactions [25,27–30]. Before one can describe how such anistropic interactions can, e.g., couple a d-wave resonance into the s wave [13], it is first necessary to efficiently characterize the resonances without such coupling.

ACKNOWLEDGMENTS

This work was partially supported by the NSF through a grant for ITAMP at Harvard University and Smithsonian Astrophysical Observatory. The work at Toledo was supported in part by the NSF under the Grant No. PHY-0758042.

APPENDIX A: REDUCTION OF AN N-CHANNEL BOUND-STATE PROBLEM TO AN EFFECTIVE $N_a < N$-CHANNEL BOUND-STATE PROBLEM

In MQDT for $-1/r^n$ type of potentials with $n > 2$ [24], the bound-states energies for an N-channel problem is given generally by the solutions of

$$\det(\chi^c - K^c) = 0, \tag{A1}$$

where K^c is an $N \times N$ real and symmetric matrix, and χ^c is an $N \times N$ diagonal matrix with elements $\chi_l^{c(n_i)}(\epsilon_{si})$.

Separating the N channels into N_a "a" channels and $N_c = N - N_a$ "c" channels, the K^c matrix can be written in a partitioned form as

$$K^c = \begin{pmatrix} K^c_{aa} & K^c_{ac} \\ K^c_{ca} & K^c_{cc} \end{pmatrix}, \tag{A2}$$

where K^c_{aa} is a $N_a \times N_a$ submatrix of K^c, K^c_{cc} is a $N_c \times N_c$ submatrix, K^c_{ac} is a $N_a \times N_c$ submatrix, and K^c_{ca} is a $N_c \times N_a$ submatrix. From $\det(xy) = \det(x)\det(y)$, we can write

$$\begin{aligned}\det(\chi^c - K^c) &= \det\begin{pmatrix} \chi^c_{aa} - K^c_{aa} & K^c_{ac} \\ K^c_{ca} & \chi^c_{cc} - K^c_{cc} \end{pmatrix} \\ &= \det\left[A\begin{pmatrix} \chi^c_{aa} - K^c_{aa} & K^c_{ac} \\ K^c_{ca} & \chi^c_{cc} - K^c_{cc} \end{pmatrix}A^{-1}B\right] \\ &= \det(A)\det\left[\begin{pmatrix} \chi^c_{aa} - K^c_{aa} & K^c_{ac} \\ K^c_{ca} & \chi^c_{cc} - K^c_{cc} \end{pmatrix}A^{-1}B\right].\end{aligned} \tag{A3}$$

Here, A is an arbitrary nonsingular matrix, and B is an arbitrary matrix with $\det(B) = 1$. Choosing

$$A = \begin{pmatrix} I & 0 \\ 0 & \chi^c_{cc} - K^c_{cc} \end{pmatrix}, \tag{A4}$$

and

$$B = \begin{pmatrix} I & 0 \\ -K^c_{ca} & I \end{pmatrix}, \tag{A5}$$

where I represents an unit matrix, we obtain

$$\det(\chi^c - K^c) = \det\left(\chi^c_{cc} - K^c_{cc}\right)\det\left(\chi^c_{aa} - K^c_{\text{eff}}\right), \tag{A6}$$

where

$$K^c_{\text{eff}} = K^c_{aa} + K^c_{ac}\left(\chi^c_{cc} - K^c_{cc}\right)^{-1}K^c_{ca}. \tag{A7}$$

Equation (A6) means that if the channels "a" and channels "c" are not coupled, namely $K^c_{ac} = 0$, the bound states separate into two sets, one for channels "a," given by $\det(\chi^c_{aa} - K^c_{aa}) = 0$, and one for channels "$c$," given by $\det(\chi^c_{cc} - K^c_{cc}) = 0$. [Since K^c, like any other K matrix, is real and symmetric, $K^c_{ac} = 0$ also implies $K^c_{ca} = (K^c_{ac})^T = 0$.] If they are coupled, which is the case that we are interested in, the solutions of $\det(\chi^c_{cc} - K^c_{cc}) = 0$ are no longer solutions of $\det(\chi^c - K^c) = 0$. This can be proven by taking the limit of $E \to \bar{E}$ in Eq. (A6), where $\bar{E}$ is one of the solutions of $\det(\chi^c_{cc} - K^c_{cc}) = 0$, namely one of the bare resonance energies. Thus, in the coupled case, all bound-state energies are given by the solutions of

$$\det\left(\chi^c_{aa} - K^c_{\text{eff}}\right) = 0, \tag{A8}$$

with the effective K^c matrix being given by Eq. (A7).

This procedure can in principle reduce an N-channel bound-state problem to an effective $N_a < N$-channel problem with N_a being an arbitrary number smaller than N. In reality, the choice of N_a is of course determined by the underlying physics. While we are using $N_a = 1$ in this paper, in which case Eq. (A8) reduces to Eq. (3), other choices are possible and are likely to be important in future treatments that incorporate the magnetic dipole-dipole and second-order spin-orbit interactions [25,27–30].

APPENDIX B: AN ALTERNATIVE PARAMETRIZATION OF MAGNETIC FESHBACH RESONANCES AND THE SPECIAL CASE OF INFINITE (GENERALIZED) BACKGROUND SCATTERING LENGTH

As stated in the main text, the parametrization adopted, Eqs. (18) for the $K_l^{c0}(\epsilon, B)$ and the corresponding Eq. (21) for the generalized scattering length, have the limitation that they fail for $K^{c0}_{\text{bg}l} = 0$, corresponding to an infinite (generalized) background scattering length $\widetilde{a}_{\text{bg}l} = \infty$. We show in this appendix that this is not an intrinsic difficulty of the theory, but due simply to the desire of using parameters that have more direct experimental interpretations. There are parametrizations of K_l^{c0} and $\widetilde{a}_l$ that would work for arbitrary background scattering lengths, provided that we are willing to sacrifice using B_{0l}.

It is not surprising that *any* parametrization based on B_{0l} would have difficulty at $K^{c0}_{\text{bg}l} = 0$ ($\widetilde{a}_{\text{bg}l} = \infty$), corresponding to having a bound or quasibound background state right at the threshold. The interaction of this background state and the "bare" Feshbach state is such that B_{0l} no longer exists, meaning that there can never be, in this case, a (coupled) bound state right at the threshold due to avoided crossing. This is reflected in the fact that the effective K_l^{c0}, given by Eq. (9), does not have a solution for $K_l^{c0}(\epsilon = 0, B_{0l}) = 0$ in the special case of $K^{c0}_{\text{bg}l} = 0$.

This difficulty can be overcome by expanding $\bar{\epsilon}_l(B)$ in Eq. (9) around $\bar{B}_{0l}$, determined by $\bar{\epsilon}_l(\bar{B}_{0l}) = 0$, which is the magnetic field at which the "bare" Feshbach resonance is crossing the threshold. This gives us $\bar{\epsilon}_l(B) \approx \delta\bar{\mu}_l(B - \bar{B}_{0l})$, where $\delta\bar{\mu}_l = d\bar{\epsilon}_l(B)/dB|_{B=\bar{B}_{0l}}$. Equation (9) now becomes

$$K_l^{c0}(\epsilon, B) = K^{c0}_{\text{bg}l} - \frac{\Gamma_l^{c0}/2}{\epsilon - \delta\bar{\mu}_l(B - \bar{B}_{0l}) - f_{El}}. \tag{B1}$$

It is a parametrization with four parameters, $K^{c0}_{\text{bg}l}$, $\bar{B}_{0l}$, $\delta\bar{\mu}_l$, and Γ_l^{c0}. [The f_{El} is not an independent parameter and is still given by Eq. (13).] The corresponding parametrization of $\widetilde{a}_l$ is

$$\widetilde{a}_l(B) = \bar{a}_l\frac{\left[1 + (-1)^l K^{c0}_{\text{bg}l}\right](B - \bar{B}_{0l} + \bar{f}_{Bl}) + (-1)^l\Gamma^{c0}_{Bl}/2}{K^{c0}_{\text{bg}l}(B - \bar{B}_{0l} + \bar{f}_{Bl}) + \Gamma^{c0}_{Bl}/2}, \tag{B2}$$

where $\bar{f}_{Bl} = f_{El}/\delta\bar{\mu}_l$ and $\Gamma^{c0}_{Bl} = \Gamma^{c0}_l/\delta\bar{\mu}_l$. Equations (B1) and (B2) work for both infinite and zero (generalized) background scattering lengths. For example, in the case of $K^{c0}_{\mathrm{bg}l} = 0$ ($\widetilde{a}_{\mathrm{bg}l} = \infty$), Eq. (B2) becomes

$$\widetilde{a}_l(B) = \bar{a}_l \left[(-1)^l + \tan(\pi\nu_0/2) + \frac{B - \bar{B}_{0l}}{\Gamma^{c0}_{Bl}/2} \right]. \quad \text{(B3)}$$

The disadvantage of this parametrization is that the parameter $\bar{B}_{0l}$ does not have as direct of an experimental interpretation as B_{0l}. It is more of a theoretical concept corresponding to the magnetic field at which a "bare" Feshbach resonance is crossing the threshold. The utility of this parametrization is, however, beyond conceptual completeness. Depending on the range of magnetic field of interest, it can be the preferred parametrization for special cases with large background scattering lengths (see, e.g., Ref. [58]) and can be used with the QDT expansion in a similar manner as the one adopted in the main text.

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